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Solving non-linear viscoelastic problems via a self-adaptive precise algorithm in time domain

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Abstract

This paper presents a new scheme of time stepping for solving non-linear viscoelastic problems with a two-level expanding technique. By expanding variables at a discretized time interval, a non-linear coupled space/time domain problem with initial and boundary values can be converted into a series of recursive linear boundary value problems, the variations of variables can be described more precisely via a self-adaptive computing procedure, and the non-linear iteration can be avoided. FEM is employed to solve recursive linear boundary value problems, and numerical comparisons are made to validate the proposed algorithm.

Keywords: Time stepping; Self-adaptive algorithm; Integral equation; Non-linear viscoelasticity; Finite element

1. Introduction

Non-linear analysis is necessary for the practical application of viscoelastic materials which do not comply with linear viscoelastic theory under some conditions (see e.g. Aklonis and Macknight, 1983; Zhou and Liu, 1996). During the past decades, a number of constitutive models of non-linear viscoelasticity have been developed, among which integral model is convenient for mathematical analysis and experimental determination of material characteristics, and is therefore widely used in practical engineering (see e.g. Bernstein et al., 1965; Findley and Onaran, 1965; Christensen, 1980; Schapery, 1969).

In addition to constitutive models, another important aspect of non-linear viscoelastic analysis is the investigation of stress and deformation. Due to the complex material properties, boundary conditions, and boundary shapes, etc., analytical solutions to non-linear viscoelastic problems are hardly obtained in general, thus a variety of numerical solutions have been presented. The deflections of viscoelastic cantilever beams are investigated by Rogers in an interative method (see e.g. Rogers and Lee, 1962), and Holden in one

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order central FD technique with a simple finite summation relation (see e.g. Holden, 1972). Henriksen made a two-dimensional FEM based nonlinear viscoelastic analysis of an isotropic material (see e.g. Henriksen, 1984), who adopted a linearized assumption which probably limits the size of time step and degrades computing accuracy. By employing Schapery's integral model, one of the most commonly applied non-linear viscoelastic models (see e.g. Beijer and Spoormaker, 2002), Tuttle and Brinson predicted non-linear viscoelastic response of laminated composites (see e.g. Tuttle and Brinson, 1986), Roy and Reddy offered a similar analysis with an additional consideration of large displacement and moisture diffusion (see e.g. Roy and Reddy, 1988a,b). Kennedy and Wang gave a fully three-dimensional non-linear viscoelastic analysis of laminated composites (see e.g. Kennedy and Wang, 1994). Touati and Cederbaum solved viscoelastic post buckling problems via a transformation between integral and differential equations (see e.g. Touati and Cederbaum, 1997). Kennedy gave a non-linear viscoelastic analysis with a layered shell element (see e.g. Kennedy, 1998). Recently, the dynamic response of structures with fractional differential operator of damping was obtained by Ingman and Suzdalnitsky in an iteration method via Laplace and Fourier transformations and series expansion (see e.g. Ingman and Suzdalnitsky, 2001). Beijer provided efficient FEM based strategies for analyzing non-linear viscoelastic polymers. (see e.g. Beijer and Spoormaker, 2002).

In the work mentioned above, the adaptability of computing accuracy to the change of sizes of time steps (see e.g. Yang, 1999a,b), which is generally difficult to be predicted in the computation, seems not to be much noted, furthermore non-linear iterations were usually required. With these considerations, a self-adaptive precise algorithm in time domain for a non-linear differential–integral equation system, without requiring iteration for the non-linear solutions, is presented in this paper. Previously this kind of algorithm was only used for the solution of differential equation system (see e.g. Yang, 1999a,b, 2000, 2001). By expending all variables at two levels at a discretized time interval, a non-linear differential–integral equation system with boundary and initial values is converted into a series of recurrent linear boundary value problems for which corresponding FEM based formulae have been derived. In the process of solving recursive FE equations, a self-adaptive computation can be carried out with different sizes of time steps. Good accordance can be observed by comparing the results given by proposed approach with an analytical solution, the solutions obtained by Houbolt time integral algorithms (see e.g. Chung and Hulbert, 1994) and a recursive method (see e.g. Henriksen, 1984).

2. Recursive governing equations

One-dimensional governing equations of dynamic viscoelastic problems can be described by (see e.g. Christensen, 1982)

$$\frac{\partial \sigma}{\partial x} + B = \rho \frac{\partial^2 u}{\partial t^2} \tag{1}$$

$$\varepsilon = \frac{\partial u}{\partial x} \tag{2}$$

The boundary conditions are specified by

$$u = \tilde{u}, \quad x \in \Gamma_u \tag{3}$$

$$\sigma = p = \tilde{p}, \quad x \in \Gamma_\sigma \tag{4}$$

where σ and ε denote the stress and strain, respectively, B is the body force, ρ designates the mass density, u is the displacement, p denotes the traction, $\tilde{u}$ and $\tilde{p}$ are the prescribed values of u and p on Γ_u and Γ_σ, respectively, $\Gamma = \Gamma_u + \Gamma_\sigma$ represents the whole boundary of the problem.

We divide time domain into a number of intervals where the initial points and sizes of time intervals are defined by $\tau_0, t_1, t_2, \ldots, t_k, \ldots$ and $T_1, T_2, \ldots, T_k, \ldots$, respectively. At a discretized time interval, in order to describe the variation of variables more precisely, all variables are expanded in the term of s

$$\sigma = \sum_{m=0} \sigma^m s^m \tag{5}$$

$$\varepsilon = \sum_{m=0} \varepsilon^m s^m \tag{6}$$

$$B = \sum_{m=0} B^m s^m \tag{7}$$

$$u = \sum_{m=0} u^m s^m \tag{8}$$

$$\tilde{u} = \sum_{m=0} \tilde{u}^m s^m \tag{9}$$

$$p = \sum_{m=0} p^m s^m \tag{10}$$

$$\tilde{p} = \sum_{m=0} \tilde{p}^m s^m \tag{11}$$

$$s = \frac{t - t_{k-1}}{T_k}$$

where t_{k-1} and T_k represent the initial point and size of kth time interval, respectively, σ^m and ε^m are the expanding coefficients of σ and ε, B^m denotes the expanding coefficient of B, u^m, p^m, $\tilde{u}^m$ and $\tilde{p}^m$ are the expanding coefficients of u, p, $\tilde{u}$ and $\tilde{p}$.

The conversion relationship of differentials from t to s is

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{T_k}\frac{\mathrm{d}}{\mathrm{d}s}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{1}{T_k^2}\frac{\mathrm{d}^2}{\mathrm{d}s^2}$$

Substitution of Eqs. (5)–(11) for Eqs. (1)–(4) then yields

$$\frac{\partial \sigma^m}{\partial x} + B^m = \frac{\rho(m+1)(m+2)}{T_k^2} u^{m+2} \tag{12}$$

$$\varepsilon^m = \frac{\partial u^m}{\partial x} \tag{13}$$

$$u^m = \tilde{u}^m, \quad \text{on } \Gamma_u \tag{14}$$

$$\sigma^m = p^m = \tilde{p}^m, \quad \text{on } \Gamma_\sigma \tag{15}$$

3. Recursive constitutive equation

A non-linear viscoelastic constitutive equation in an integral form can be described by (see e.g Zhou and Liu, 1996)

$$\varepsilon(t) = \frac{\sigma(t)}{E(t)} - \int_{\tau_0}^{t} \sigma(\tau) \frac{\partial}{\partial \tau} \left[\frac{1}{E(\tau)} \right] \mathrm{d}\tau - \int_{\tau_0}^{t} f[\sigma(\tau)] \frac{\partial}{\partial \tau} C(t,\tau)\, \mathrm{d}\tau \tag{16}$$

where $E(t)$ denotes Young's modulus, τ_0 is the lower limit of integration.

$$1/E(t) = A(t) = a + h\mathrm{e}^{-bt} \tag{17}$$

$$f(\sigma) = \alpha\sigma + \lambda\sigma^2 \tag{18}$$

$$C(t,\tau) = \varphi(\tau)(1 - \mathrm{e}^{-\gamma(t-\tau)}) + \psi(\tau) - \psi(t) \tag{19}$$

$$\varphi(\tau) = c + d\mathrm{e}^{-b\tau} \tag{20}$$

$$\psi(\tau) = q\mathrm{e}^{-g\tau} \tag{21}$$

a, b, c, d, q, g and γ are material data to be given.

Two kinds of variables are involved in Eqs. (16)–(21), and related to τ and t, respectively. Thus a two-level expanding technique is adopted. One level is for variables concerned with τ in the integration via an expanding variable ξ, another level is for variables relevant to t, via an expanding variable s.

At the first level, $\sigma(\tau)$, $A(\tau)$, $\varphi(\tau)$ and $\psi(\tau)$ are expanded as

$$\sigma(\xi) = \sum_{m=0} \sigma^m \xi^m \tag{22}$$

$$A(\xi) = a + h\mathrm{e}^{-b(T_k\xi + t_{k-1})} = a + h\mathrm{e}^{-bt_{k-1}}\mathrm{e}^{-bT_k\xi} = a + h\mathrm{e}^{-bt_{k-1}} \sum_{m=0} \left(\frac{(-bT_k)^m}{m!} \right) \xi^m = \sum_{m=0} A^m \xi^m \tag{23}$$

$$A^0 = a + h\mathrm{e}^{-bt_{k-1}}, \quad A^m = h\mathrm{e}^{-bt_{k-1}} \frac{(-bT_k)^m}{m!}, \quad m = 1, 2, 3, \ldots$$

$$\varphi(\xi) = c + d\mathrm{e}^{-b(T_k\xi + t_{k-1})} = c + d\mathrm{e}^{-bt_{k-1}} \sum_{m=0} \left(\frac{(-bT_k)^m}{m!} \right) \xi^m = \sum_{m=0} \varphi^m \xi^m \tag{24}$$

$$\varphi^0 = c + d\mathrm{e}^{-bt_{k-1}}, \quad \varphi^m = d\mathrm{e}^{-bt_{k-1}} \frac{(-bT_k)^m}{m!}, \quad m = 1, 2, 3, \ldots$$

$$\psi(\xi) = q\mathrm{e}^{-g(T_k\xi + t_{k-1})} = q\mathrm{e}^{-gt_{k-1}} \sum_{m=0} \frac{(-gT_k)^m}{m!} \xi^m = \sum_{m=0} \psi^m \xi^m \tag{25}$$

$$\psi^m = q\mathrm{e}^{-gt_{k-1}} \frac{(-gT_k)^m}{m!}, \quad m = 0, 1, 2, \ldots$$

$$\mathrm{e}^{\gamma T_k \xi} = \sum_{m=0} \frac{(T_k\gamma)^m}{m!} \xi^m = \sum_{m=0} e_2^m \xi^m \tag{26}$$

$$e_2^m = \frac{(T_k\gamma)^m}{m!}, \quad m = 0, 1, 2, \ldots$$

where A^m, φ^m, ψ^m, e_2^m denote the expanding coefficients of $A(\xi)$, $\varphi(\xi)$, $\psi(\xi)$ and $e^{T\gamma\xi}$.

$$\xi = \frac{\tau - t_{k-1}}{T_k}, \quad \xi \in [0, s].$$

The conversion relationship of differentials from τ to ξ is

$$\frac{\partial}{\partial\tau} = \frac{\partial}{\partial\xi}\frac{\partial\xi}{\partial\tau} = \frac{1}{T_k}\frac{\partial}{\partial\xi}$$

At the second level, $\sigma(t)$ and $\varepsilon(t)$ in Eq. (16) are expanded by Eqs. (5) and (6), $A(t)$ is expanded by

$$A(s) = \sum_{m=0} A'^m s^m \tag{27}$$

$$A'^m = A^m, \quad m = 0, 1, 2, \ldots$$

$e^{-\gamma t}$, decomposed from $e^{-\gamma(t-\tau)}$ with $e^{\gamma\tau}$ in Eq. (19), is expanded by

$$e^{-\gamma T_k s} = \sum_{m=0} \frac{(-T_k\gamma)^m}{m!} s^m = \sum_{m=0} e_1^m s^m \tag{28}$$

$$e_1^m = \frac{(-T_k\gamma)^m}{m!}, \quad m = 0, 1, 2, \ldots$$

where e_1^m denotes the expanding coefficient of $e^{-T\gamma s}$.

At the first time interval where $t \in [\tau_0, \tau_0 + T_1]$, substituting Eqs. (5) and (6) and Eqs. (22)–(28) into Eq. (16) can yield

$$\begin{aligned}
\sum_{r=0} \varepsilon^r s^r = {} & \sum_{r=0} \left(\sum_{m=0}^{r} \sigma^m A^{r-m} \right) s^r - \int_0^s \sum_{r=0} \left(\sum_{m=0}^{r} (m+1) A^{m+1} \sigma^{r-m} \right) \xi^r \, \mathrm{d}\xi \\
& - \alpha \int_0^s \sum_{r=0} \left(\sum_{m=0}^{r} (m+1) \varphi^{m+1} \sigma^{r-m} \right) \xi^r \, \mathrm{d}\xi + \alpha \sum_{m=0} e_1^m s^m \int_0^s \sum_{k=0} \left(\sum_{n=0}^{k} (n+1) \left(\sum_{m=0}^{n} \varphi^m e_2^{n-m+1} \right) \sigma^{k-n} \right) \xi^k \, \mathrm{d}\xi \\
& - \alpha \int_0^s \sum_{r=0} \left(\sum_{m=0}^{r} (m+1) \psi^{m+1} \sigma^{r-m} \right) \xi^r \, \mathrm{d}\xi - \lambda \int_0^s \sum_{r=0} \left(\sum_{n=0}^{r} \left(\sum_{m=0}^{n} (m+1) \varphi^{m+1} \sigma^{n-m} \right) \sigma^{r-n} \right) \xi^r \, \mathrm{d}\xi \\
& + \lambda \sum_{m=0} e_1^m s^m \int_0^s \sum_{l=0} \left(\sum_{k=0}^{l} \left(\sum_{n=0}^{k} (n+1) \left(\sum_{m=0}^{n+1} \varphi^m e_2^{n-m+1} \right) \sigma^{k-n} \right) \sigma^{l-k} \right) \xi^l \, \mathrm{d}\xi \\
& - \lambda \int_0^s \sum_{r=0} \left(\sum_{n=0}^{r} \left(\sum_{m=0}^{n} (m+1) \psi^{m+1} \sigma^{n-m} \right) \sigma^{r-n} \right) \xi^r \, \mathrm{d}\xi
\end{aligned} \tag{29}$$

Integrating right-hand side of Eq. (29) with respect to ξ then gives

$$\begin{aligned}\sum_{r=0}^{\infty}\varepsilon^r s^r = {} & \sum_{r=0}^{\infty}\left(\sum_{m=0}^{r}\sigma^m A^{r-m}\right)s^r - \sum_{r=1}^{\infty}\frac{1}{r}\left(\sum_{m=0}^{r-1}(m+1)A^{m+1}\sigma^{r-m-1}\right)s^r \\ & - \alpha\sum_{r=1}^{\infty}\frac{1}{r}\left(\sum_{m=0}^{r-1}(m+1)\varphi^{m+1}\sigma^{r-m-1}\right)s^r \\ & + \alpha\sum_{r=1}^{\infty}\left(\sum_{k=0}^{r-1}\frac{1}{k+1}\left(\sum_{n=0}^{k}(n+1)\left(\sum_{m=0}^{n+1}\varphi^m e_2^{n-m+1}\right)\sigma^{k-n}\right)e_1^{r-k-1}\right)s^r \\ & - \alpha\sum_{r=1}^{\infty}\frac{1}{r}\left(\sum_{m=0}^{r-1}(m+1)\psi^{m+1}\sigma^{r-m-1}\right)s^r - \lambda\sum_{r=1}^{\infty}\frac{1}{r}\left(\sum_{n=0}^{r-1}\left(\sum_{m=0}^{n}(m+1)\varphi^{m+1}\sigma^{n-m}\right)\sigma^{r-n-1}\right)s^r \\ & + \lambda\sum_{r=1}^{\infty}\left(\sum_{l=0}^{r-1}\frac{1}{l+1}\left(\sum_{k=0}^{l}\left(\sum_{n=0}^{k}(n+1)\left(\sum_{m=0}^{n+1}\varphi^m e_2^{n-m+1}\right)\sigma^{k-n}\right)\sigma^{l-k}\right)e_1^{r-l-1}\right)s^r \\ & - \lambda\sum_{r=1}^{\infty}\frac{1}{r}\left(\sum_{n=0}^{r-1}\left(\sum_{m=0}^{n}(m+1)\psi^{m+1}\sigma^{n-m}\right)\sigma^{r-n-1}\right)s^r \end{aligned} \tag{30}$$

where r denotes the power of expanding variable s.

Equating the powers on the two sides of Eq. (30) then yields

$$\varepsilon^0 = \sigma^0 A^0, \quad r = 0 \tag{31}$$

$$\sigma^r A^0 = \varepsilon^r - \mathrm{ET}(r), \quad r \neq 0 \tag{32}$$

where

$$\begin{aligned}\mathrm{ET}(r) = {} & \sum_{m=0}^{r-1}\sigma^m A^{r-m} - \frac{1}{r}\sum_{m=0}^{r-1}(m+1)A^{m+1}\sigma^{r-m-1} - \alpha\frac{1}{r}\sum_{m=0}^{r-1}(m+1)\varphi^{m+1}\sigma^{r-m-1} \\ & + \alpha\sum_{k=0}^{r-1}\frac{1}{k+1}\left(\sum_{n=0}^{k}(n+1)\left(\sum_{m=0}^{n+1}\varphi^m e_2^{n-m+1}\right)\sigma^{k-n}\right)e_1^{r-k-1} \\ & - \alpha\frac{1}{r}\sum_{m=0}^{r-1}(m+1)\psi^{m+1}\sigma^{r-m-1} - \lambda\frac{1}{r}\sum_{n=0}^{r-1}\left(\sum_{m=0}^{n}(m+1)\varphi^{m+1}\sigma^{n-m}\right)\sigma^{r-n-1} \\ & + \lambda\sum_{l=0}^{r-1}\frac{1}{l+1}\left(\sum_{k=0}^{l}\left(\sum_{n=0}^{k}(n+1)\left(\sum_{m=0}^{n+1}\varphi^m e_2^{n-m+1}\right)\sigma^{k-n}\right)\sigma^{l-k}\right)e_1^{r-l-1} \\ & - \lambda\frac{1}{r}\sum_{n=0}^{r-1}\left(\sum_{m=0}^{n}(m+1)\psi^{m+1}\sigma^{n-m}\right)\sigma^{r-n-1} \end{aligned} \tag{33}$$

t_{k-1} and T_k in A^m, φ^m, ψ^m, e_1^m and e_2^m in Eqs. (23)–(28) are replaced by τ_0 and T_1, respectively.

Assuming that the solution from τ_0 to t_{k-1} has been obtained, we consider the solution from t_{k-1} to t, at the interval where $t \in [t_{k-1}, t_{k-1} + T_k]$, and re-arrange Eq. (16) in the form

$$\varepsilon(t) = \sigma(t)A(t) - C1 - \int_{t_{k-1}}^{t} \sigma(\tau)\frac{\partial}{\partial\tau}A(\tau)\,\mathrm{d}\tau - C2 + \alpha \mathrm{e}^{-\gamma t}C3 + \lambda \mathrm{e}^{-\gamma t}C4 - \int_{t_{k-1}}^{t} (\alpha\sigma(\tau) + \lambda\sigma^2(\tau))\frac{\partial}{\partial\tau}[\varphi(\tau) - \varphi(\tau)\mathrm{e}^{-\gamma(t-\tau)} + \psi(\tau) - \psi(t)]\,\mathrm{d}\tau \tag{34}$$

where

$$C1 = \int_{\tau_0}^{t_{k-1}} \sigma(\tau)\frac{\partial}{\partial\tau}A(\tau)\,\mathrm{d}\tau \tag{35}$$

$$C2 = \int_{\tau_0}^{t_{k-1}} (\alpha\sigma(\tau) + \lambda\sigma^2(\tau))\frac{\partial}{\partial\tau}[\varphi(\tau) + \psi(\tau) - \psi(t)]\,\mathrm{d}\tau \tag{36}$$

$$C3 = \int_{\tau_0}^{t_{k-1}} \sigma(\tau)\frac{\partial}{\partial\tau}[\varphi(\tau)\mathrm{e}^{\gamma\tau}]\,\mathrm{d}\tau \tag{37}$$

$$C4 = \int_{\tau_0}^{t_{k-1}} \sigma(\tau)\frac{\partial}{\partial\tau}[\varphi(\tau)\mathrm{e}^{\gamma\tau}]\,\mathrm{d}\tau \tag{38}$$

$C1$, $C2$, $C3$, $C4$ can be obtained via the solution from τ_0 to t_{k-1}.

Using the similar expanding technique adopted above, recursive equations of Eq. (34) can be obtained

$$\sigma^r A^0 = \varepsilon^r - \mathrm{EPT}(r), \quad r \neq 0 \tag{39}$$

$$\mathrm{EPT}(r) = \mathrm{ET}(r) + \alpha \mathrm{e}^{-\gamma t_{k-1}} e_1^r C3 + \lambda \mathrm{e}^{-\gamma t_{k-1}} e_1^r C4 \tag{40}$$

4. Implementation of FEM

For Eqs. (12)–(15), the framework of a FEM based solution can be established by utilizing a conventional weighted residual technique (see e.g. Zienkiewicz and Morgan, 1983), having the form

$$\int_0^l u^* \left[\frac{\partial\sigma^m}{\partial x} + B^m - \frac{\rho(m+1)(m+2)}{T^2}u^{m+2}\right]A_S\,\mathrm{d}x + A_S p^*(u^m - \tilde{u}^m)|_{\Gamma_u} - A_S u^* - (p^m - \tilde{p}^m)|_{\Gamma_\sigma} = 0 \tag{41}$$

where u^* and p^* denote weighting functions, and A_S represents the area of cross section.

By utilizing the theorem of integral by part, Eq. (41) can be written as

$$\int_0^l u^* \left[B^m - \frac{\rho(m+1)(m+2)}{T^2}u^{m+2}\right]\mathrm{d}x + u^* p^m\Big|_{\Gamma_u} - \int_0^l \frac{\partial u^*}{\partial x}\sigma^m\,\mathrm{d}x + u^*\tilde{p}^m\Big|_{\Gamma_\sigma} = 0 \tag{42}$$

In the implementation of FEM, u^m is evaluated in the terms of their nodal values at the element level, having the form

$$u^m = \underline{N}\bar{u}^m \tag{43}$$

where $\underline{N}$ represents a matrix of shape function, and $\bar{u}^m$ denotes the nodal vector of u^m.

Weighting function u^* can be described in the same form

$$u^* = \underline{N}\bar{u}^* \tag{44}$$

where $\bar{u}^*$ represent the nodal vector of u^*.

Substitution of Eqs. (43) and (44) for (42) then yields

$$\frac{(m+1)(m+2)}{T_m^2}[M]\{\bar{u}\}^{m+2} = \{\bar{p}\}^m + \{\overline{B}\}^m - \sum_{e=1}\int_{\Omega_e} N'\{\sigma\}^m dv \tag{45}$$

where

$$[M] = \sum_{e=1}\int_{\Omega_e} \underline{N}^T \rho N \,\mathrm{d}x \tag{46}$$

$$\{\overline{B}\}^m = \sum \int \underline{N}^T \{\overline{B}^m\}\,\mathrm{d}x \tag{47}$$

$$\{\bar{p}\}^m = \sum_{e=1} \underline{N}^T \{\bar{p}^m\} \tag{48}$$

$\underline{N}'$ represents a matrix of derivatives of $\underline{N}$.

At the first time interval where $t \in [\tau_0, \tau_0 + T_1]$, substitution of Eqs. (31) and (32) for Eq. (45) gives

$$\frac{2}{T_1^2}[M]\{\bar{u}\}^2 = \{\overline{B}\}^0 + \{\bar{p}\}^0 - \frac{1}{A^0}\sum_{e=1}\int_{\Omega_e} N'^T N'\{\bar{u}\}^0\,\mathrm{d}x \tag{49}$$

$$\frac{(r+1)(r+2)}{T_1^2}[M]\{\bar{u}\}^{r+2} = \{\overline{B}\}^r + \{\bar{p}\}^r + \frac{1}{A^0}\sum_{e=1}\int_{\Omega_e} N'^T \mathrm{ET}(r)\,\mathrm{d}x - \frac{1}{A^0}\sum_{e=1}\int_{\Omega_e} N'^T N'\{\bar{u}\}^r\,\mathrm{d}x \tag{50}$$

where $\{\bar{u}\}^m$ denotes the general nodal vector of u^m.

At the time interval where $t \in [t_{k-1}, t_{k-1} + T_k]$, substitution of Eq. (39) for Eq. (45) then yields

$$\frac{2}{T_{k+1}^2}[M]\{\bar{u}\}^2 = \{\overline{B}\}^0 + \{\bar{p}\}^0 - \frac{1}{A^0}\sum_{e=1}\int_{\Omega_e} N'^T N'\{\bar{u}\}^0\,\mathrm{d}x \tag{51}$$

$$\frac{(r+1)(r+2)}{T_{k+1}^2}[M]\{\bar{u}\}^{r+2} = \{\overline{B}\}^r + \{\bar{p}\}^r + \frac{1}{A^0}\sum_{e=1}\int_{\Omega_e} N'^T \mathrm{EPT}(r)\,\mathrm{d}x - \frac{1}{A^0}\sum_{e=1}\int_{\Omega_e} N'^T N'\{\bar{u}\}^r\,\mathrm{d}x \tag{52}$$

At the first time interval, $\{\bar{u}\}^0$ and $\{\bar{u}\}^1$ are provided by initial conditions, at the kth time interval, they will be given by

$$\{\bar{u}\}_{s=1} = \sum_{m=0}\{\bar{u}\}^m \quad \text{(in the } (k-1)\text{th time interval)} \tag{53}$$

$$\{\bar{u}_{,t}\}_{s=1} = \sum_{m=0}\frac{(m+1)}{T_{k-1}}\{\bar{u}\}^{m+1} \quad \text{(in the } (k-1)\text{th time interval)} \tag{54}$$

By utilizing Eqs. (49)–(54) with initial conditions, the problem defined by Eq. (1)–(4) and (16) can be solved step by step, and non-linear iteration can be avoided.

A self-adaptive computation is carried out at each time interval with a convergence criterion

$$\mathrm{Abs}\left(\left(\bar{u}_k^m s^m \Big/ \sum_{j=0}^{m-1}\bar{u}_k^j s^j\right)_{s=1}\right) \leqslant \beta \tag{55}$$

where β is an error bound, $\bar{u}_k^j$ denotes kth component of $\{\bar{u}\}^j$ $(j = 1, 2, \ldots, r)$.

Every $\{\bar{u}\}^m$ $(m = 1, 2, \ldots)$ is required to be checked with the above criterion, if the criteria is satisfied continually 3 times, computing will stop at the time interval considered, and step into next one. If the criteria is not met, $(m+1)$th recursive computation will continue till the convergence.

In the computation, mm, a upper bound of m, will be prescribed previously. If computing can not stop when $m = mm$, it is necessary to reduce the size of time step or increase mm; if condition (55) is satisfied when $m \ll mm$, a bigger size of time step can be considered. At each of discretized time intervals, truncation error depends on β, and can be estimated by K_β which is the highest power when criteria (55) is satisfied, the order of truncation error is therefore $K_\beta + 1$.

All the above procedure can be realized by a program automatically.

5. Numerical examples

5.1. Linear dynamic viscoelastic problem

Consider the motion of a mass attached to a massless viscoelastic rod as shown in Fig. 1 where the governing equation of u is specified by

$$M\ddot{u} + A\sigma = P$$

The relationship of displacement and strain is described by $\varepsilon = u/L$.

Constitutive equation is a Kelvin model that can be written in an integral form

$$\varepsilon(t) = \mathrm{e}^{-\gamma t} \int_0^t \mathrm{e}^{\gamma\tau}\sigma/q_1 \,\mathrm{d}\tau, \quad (\gamma q_0/q_1)$$

or in a differential form

$$\sigma = q_0\varepsilon + q_1\dot{\varepsilon}$$

where $M = 100$, $A = 0.02$, $L = 2$, $q_0 = 60,000$, $q_1 = 50,000$.

Table 1 exhibits the comparison between the solution obtained by the proposed algorithm for the differential–integral system and those given by the scheme for the differential systems, and an analytical solution (see e.g. Yang et al., 2001).

5.2. Non-linear static viscoelastic problem

This example considers the static analysis of a viscoelastic round rod as shown in Fig. 2 where $l = 200$ cm, $d = 20$ cm, $\rho = 0.01$ kg/cm^3. The rod is subjected to a constant extension force $P = 31415.9$ kg. The

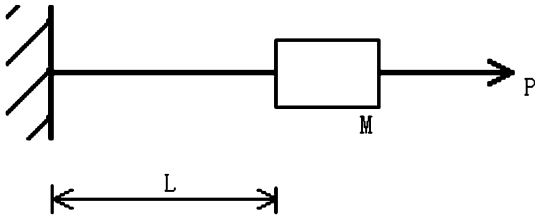

Fig. 1. A mass attached to a massless viscoelastic rod.

Table 1
Numerical comparison of a linear dynamic viscoelastic problem with Kelvin model

	β	t	u (Integral model)	u (Differential model)	u (Analytical solution)
Uniform time step = 0.2		0.2	2.440607E−03	2.440599E−03	2.440603E−03
$P = \sin(t)$		0.4	3.028552E−03	3.028516E−03	3.028514E−03
Initial displacement = 0.0	0.0001	0.6	2.894332E−03	2.894302E−03	2.894300E−03
Initial velocity = 0.02		0.8	2.557278E−03	2.557295E−03	2.557293E−03
		1.0	2.232911E−03	2.233016E−03	2.233015E−03

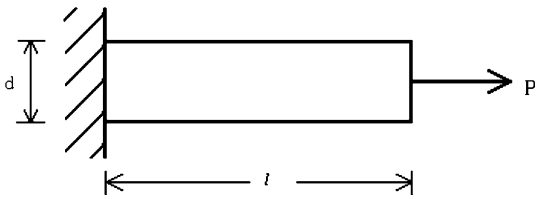

Fig. 2. A non-linear viscoelastic rod.

constitutive relationship is described by Eq. (16) where $\gamma = 3.2407 \times 10^{-6}$/s, $a = 0.27 \times 10^{-6}$, $b = 0.015$, $c = 0.8 \times 10^{-6}$, $d = 0.2 \times 10^{-6}$, $q = 0.61 \times 10^{-5}$, $g = 0.005$, $h = 0.38a$, $\alpha + \lambda = 1$.

Numerical comparison is exhibited in Tables 2 and 3 and Fig. 3 where the analytical solution is given by (see e.g. Appendix A)

$$\sigma = 100 \text{ kg/cm}^2, \quad u(t,x) = (\sigma A(\tau_0) + (\alpha\sigma + \lambda\sigma^2)C(t,\tau_0))x$$

Henriksen's method in Tables 2 and 3 refers to the work given by Henriksen (see e.g. Henriksen, 1984).

Table 2
Numerical comparison of a linear static viscoelastic problem ($\alpha = 1.0, \lambda = 0.0$)

	β	t	$u\|_{x=l}$ (Proposed algorithm)	$u\|_{x=l}$ (Analytical solution)	$u\|_{x=l}$ (Henriksen's method)
Uniform time step = 0.2		0.2	0.068603118987226	0.068603118987226	0.068603118987226
$P = 31415.9$ kg		0.4	0.069668277991610	0.069668277991610	0.069668277991610
	0.000001	0.6	0.070681089219966	0.070681089219967	0.070681089219967
		0.8	0.071644397918591	0.071644397918591	0.071644397918591
		1.0	0.072560894384859	0.072560894384859	0.072560894384859

Table 3
Numerical comparison of a non-linear static viscoelastic problem ($\alpha = 0.995, \lambda = 0.005$)

	β	t	$u\|_{x=l}$ (Proposed algorithm)	$u\|_{x=l}$ (Analytical solution)	$u\|_{x=l}$ (Henriksen's method)
Uniform time step = 0.2		0.2	0.06915777383011	0.06915777382997	0.06915777382892
$P = 31415.9$ kg		0.4	0.07075018604001	0.07075018603969	0.07075018603765
	0.000001	0.6	0.07226433834946	0.07226433834890	0.07226433834593
		0.8	0.07370448440033	0.07370448439948	0.07370448439562
		1.0	0.07507464618591	0.07507464618474	0.07507464618004

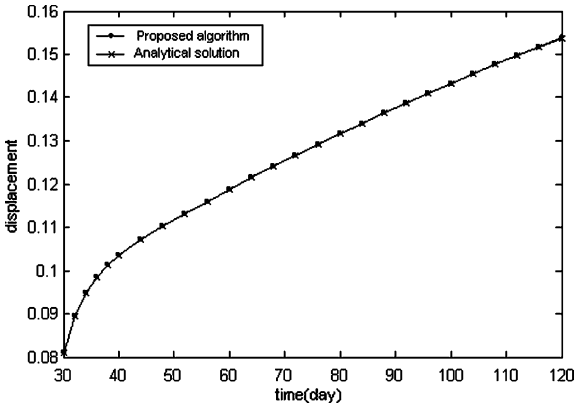

Fig. 3. Numerical comparison of non-linear static solutions.

5.3. Non-linear dynamic viscoelastic problem

This example illustrates free and forced vibrations of a non-linear viscoelastic rod shown in Fig. 2 where computing parameters are the same as those used in the Example 5.2.

Numerical comparison is given in Tables 4 and 5, and Figs. 4–6 where it can be observed that the precision of presented self-adaptive algorithm has not been affected by the change of size of time steps, whereas when size of time step = 0.07s, notable degradation of precision can be seen in the Houbolt algorithm based solution (see e.g. Chung and Hulbert, 1994), as shown in Fig. 6.

Table 4
Numerical comparison of free vibration of a linear viscoelastic dynamic problem ($\alpha = 1.0, \lambda = 0.0$)

t	$u\|_{x=l}$ $\beta = 0.0000001$ (Proposed algorithm) Uniform time step = 0.01	$u\|_{x=l}$ $\beta = 0.0000001$ (Proposed algorithm) Uniform time step = 0.05	$u\|_{x=l}$ (Houbolt algorithm) Uniform time step = 0.0001	$u\|_{x=l}$ (Houbolt algorithm) Uniform time step = 0.001	$u\|_{x=l}$ (Houbolt algorithm) Uniform time step = 0.05
0.05	0.20991797	0.20991724	0.20991770	0.20991755	0.20713907
0.10	0.19856094	0.19854581	0.19856008	0.19855984	0.19514576
0.15	0.18811936	0.18808918	0.18811849	0.18811853	0.18779845
0.20	0.18144076	0.18141607	0.18144088	0.18144123	0.18344093
0.25	0.17553374	0.17552580	0.17553461	0.17553476	0.17581971

Table 5
Numerical comparison of free vibration of a non-linear dynamic viscoelastic problem ($\alpha = 0.995, \lambda = 0.005$)

t	$u\|_{x=l}$ (Proposed algorithm) Uniform time step = 0.01	$u\|_{x=l}$ (Proposed algorithm) Uniform time step = 0.05	$u\|_{x=l}$ (Houbolt algorithm) Uniform time step = 0.0001	$u\|_{x=l}$ (Houbolt algorithm) Uniform time step = 0.001	$u\|_{x=l}$ (Houbolt algorithm) Uniform time step = 0.05
0.05	0.20992894	0.20992797	0.20992842	0.20992829	0.20718364
0.10	0.19862666	0.19860527	0.19861920	0.19861897	0.19521788
0.15	0.18827681	0.18821127	0.18823989	0.18823992	0.18789230
0.20	0.18168778	0.18160691	0.18163111	0.18163147	0.18363860
0.25	0.17587062	0.17582863	0.17583715	0.17583732	0.17619707

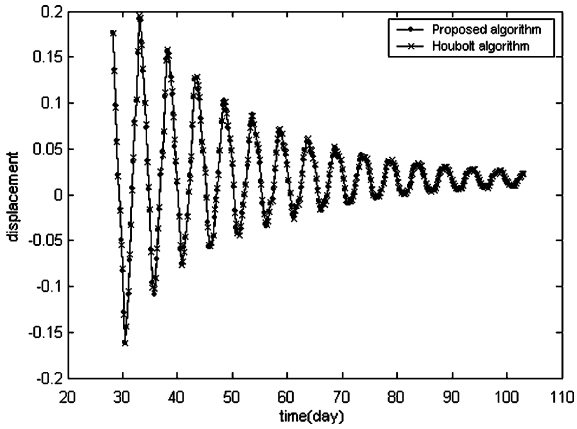

Fig. 4. Numerical comparison of free non-linear vibration ($u(l) = 0.215$ cm, $\dot{u}(l) = 0.0$).

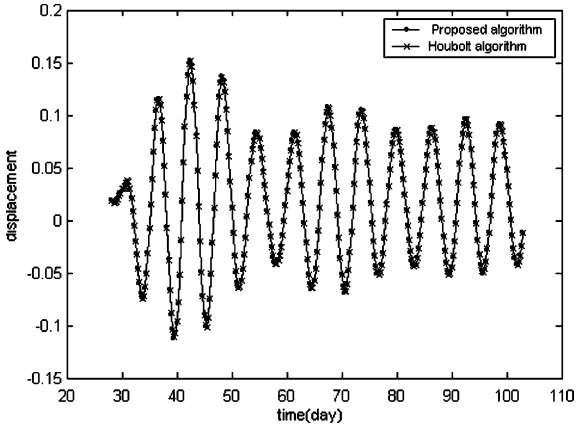

Fig. 5. Numerical comparison of forced non-linear vibration ($P(t) = 50A\sin(t)$ kg, $\dot{u}(l) = 0.0$, $u(l) = 0.0215$ cm).

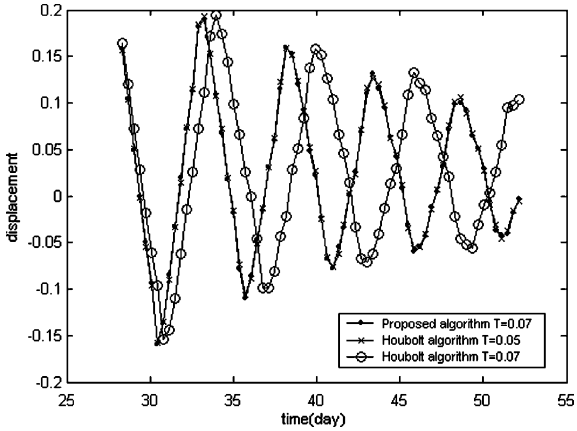

Fig. 6. Comparison of the precision of non-linear free vibration ($u(l) = 0.215$ cm, $\dot{u}(l) = 0.0$).

For the solution of free vibration, Figs. 7 and 8 show the variations of recursion steps in the whole self-adaptive computing process. It is obvious that recursion steps increase with the expansion of sizes of time

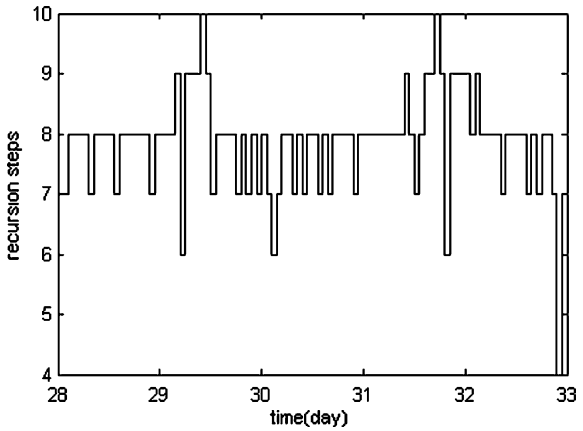

Fig. 7. The variation of recursion steps in the computing for non-linear free vibration with uniform time step = 0.05.

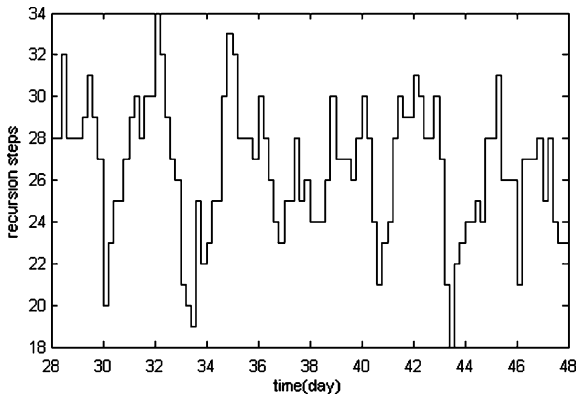

Fig. 8. The variation of recursion steps in the computing for non-linear free vibration with uniform time step = 0.2.

Table 6
Numerical comparison of free vibration with non-uniform and uniform sizes of time steps

t	β	$u\|_{x=l}$ (Proposed algorithm) Variational sizes of time steps		$u\|_{x=l}$ (Proposed algorithm) Uniform time step = 0.02
0.2		Size of time step = 0.02	0.181634327	0.181634327
0.9		Size of time step = 0.04	0.072620927	0.072602554
1.3		Size of time step = 0.04	0.012820230	0.012819433
2.3	0.000001	Size of time step = 0.08	−0.137571746	−0.137567538
3.1		Size of time step = 0.08	−0.091271220	−0.091312303
5.1		Size of time step = 0.16	0.194604353	0.194590394
6.7		Size of time step = 0.16	0.118072247	0.117846597

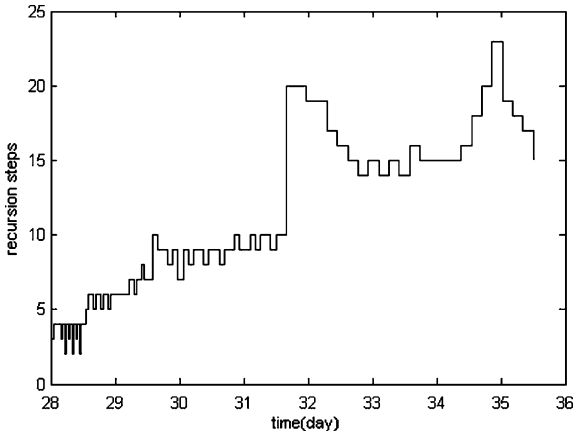

Fig. 9. The variation of recursion steps in the computing for non-linear free vibration with non-uniform sizes of time steps.

steps. In Table 6, a satisfactory comparison of solutions with non-uniform and uniform sizes of time steps is demonstrated. The variations of recursion steps are shown in Fig. 9.

6. Conclusions

This paper presents a new algorithm of time stepping for solving non-linear viscoelastic problems, the merits of this algorithm include

1. A coupled differential and integral equation system with initial and boundary values is converted into a series of recurrent linear boundary value problems which are solved by FEM as in the case of static elasticity. In addition to EFM, other well developed numerical approaches can also be employed with regard to the characteristics of the problem, including in-homogeneity, complex boundary geometry, and boundary conditions, etc.
2. Self-adaptive computation can provide a more precise description for the variation of variables, and compensate for the possible loss of computing accuracy caused by improper choices of the size of time step.
3. For non-linear cases, no assumption is ever made, and no iteration is ever needed.

Due to the good performance of numerical validation, the proposed method can hopefully be utilized to solve more complicated problems.

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Appendix A

Eq. (1) is met by substituting $\sigma = 100$ kg/cm^2 into Eq. (1) in the static case with $B = 0$. For Eq. (4), one has

$$\sigma_{x=L} = 4P/\pi d^2 = 100 \text{ kg/cm}^2$$

Substituting $\sigma = 100$ kg/cm^2 into Eq. (16) gives

$$\varepsilon(t) = \sigma A(\tau_0) + (\alpha\sigma + \lambda\sigma^2)C(t, \tau_0) \tag{A.1}$$

Integrating Eq. (A.1) with $u|_{x=0} = 0$ then yields

$$u(t, x) = x\{\sigma A(\tau_0) + (\alpha\sigma + \lambda\sigma^2)C(t, \tau_0)\}$$

Therefore Eq. (1)–(4), and (16) are all satisfied.

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